ON THE IMPACT OF A VISCO-PLASTIC BAR ON A RIGID OBSTACLE

(OB UDARE VIAZKO-PLASTICHESKOGO STERZHNIA O ZHESTKUIU PREGRADU)

PMM Vol. 26, No. 3, 1962, pp. 497-502

G.I. BARENBLATT and A.Iu. ISHLINSKII
(Moscow)

(Received February 15, 1962)

The problem of the unsteady motion of a visco-plastic body has attracted for some time the attention of research workers [1-3]. The analysis of available exact and approximate solutions of unsteady problems has been given in a monograph by Mirzadzhanzade [4].

In this paper a formulation and effective approximate solution will be given of the problem concerning the impact on the rigid obstacle of a visco-plastic bar of finite length. The problem of the elasto-plastic impact of a bar on a solid obstacle was considered by Lenskii [5].

1. Formulation of the problem. A bar of finite length, consisting of visco-plastic incompressible material, is translated in the direction of its longitudinal axis and with an initial velocity $-v_0$ at time $t = 0$ strikes a solid obstacle (Fig. 1).

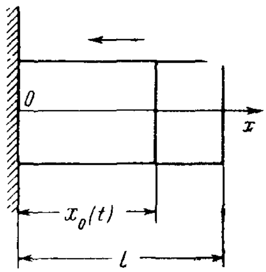

Fig. 1.

We assume that the motion of the bar is almost uniform, i.e. the stress, the velocity, etc., are given as the average value over the section of the bar.

In the given case the relation between the values, averaged over the cross-section, of the stress σ and the velocity of deformation $\partial v/\partial x$ in the visco-plastic medium are given by

$$\frac{\partial v}{\partial x} = \begin{cases} \dfrac{\sigma + \sigma_0}{\mu} & (|\sigma| \geqslant \sigma_0) \\ 0 & (|\sigma| \leqslant \sigma_0) \end{cases} \qquad (1.1)$$

where $v(x, t)$ is the velocity of the section of the bar at time t; $\sigma_0 > 0$ is the stress at the limit point; μ is the coefficient of the viscosity of the materials of the bar, and the coordinate x is directed along the axis of the bar and is oriented opposite to the direction of the motion; clearly, $\sigma \leqslant 0$ at all points.

Physically it is evident that the pattern of motion has the following form. Taking into account that the propagation velocity of the elastic disturbance in the considered medium is very large, because the Young's modulus of that medium is large, the disturbance takes place almost instantaneously over the whole bar. Then, the velocity of the motion for an arbitrary $t > 0$ differs from $-v_0$ at all points of the bar.

The bar will be divided into two parts. In one part $(0 \leqslant x \leqslant x_0(t))$, which can be called the *visco-plastic region*, the stresses exceed σ_0 and visco-plastic flow is obtained. In the second part $(x_0(t) \leqslant x \leqslant l)$ which we call the elastic (rigid) region the stress is less than σ_0, so that this part of the bar is moving as a rigid body. On the moving boundary between the visco-plastic and the elastic part $x = x_0(t)$, whose position has to be determined in the course of the solution of problem, stress and velocity are continuous.

The fundamental equation of motion has the following form

$$\rho \frac{\partial v}{\partial t} = \frac{\partial \sigma}{\partial x} \tag{1.2}$$

where ρ is the density of the material of the bar, which we assume to be constant; t is time. Then, by virtue of Equation (1.1) in the visco-plastic region the velocity satisfies the heat equation

$$\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2}, \qquad a^2 = \frac{\mu}{\rho} \qquad (0 \leqslant x < x_0(t)) \tag{1.3}$$

and in the elastic region the equation

$$\frac{\partial v}{\partial x} = 0 \qquad (x_0(t) \leqslant x \leqslant l) \tag{1.4}$$

After integrating Equation (1.4), it follows

$$v = -v_0(t) \qquad (x_0(t) \leqslant x \leqslant l) \tag{1.5}$$

where $-v_0(t)$ represents the motion of the elastic region of the bar, which is an unknown function of time.

Equation of motion in the elastic region is given by

$$M \frac{dv_0(t)}{dt} = \rho F_0 [l - x_0(t)] \frac{dv_0(t)}{dt} = \sigma [x_0(t) + 0, t] F_0 \tag{1.6}$$

where M is the mass of the elastic part of the bar and F_0 is the area of the cross-section of the bar.

Taking into account that the stress on the moving boundary, $x = x_0(t)$, is continuous, the relation (1.6) leads to

$$\frac{dv_0(t)}{dt} = -\frac{\sigma_0}{\rho\,[l - x_0(t)]} \tag{1.7}$$

Furthermore, by virtue of the continuity of the velocity on the moving boundary, $x = x_0(t)$, we have

$$v\,[x_0(t), t] = -v_0(t), \qquad \frac{\partial}{\partial x} v\,[x_0(t), t] = 0 \tag{1.8}$$

The boundary and initial conditions are given by

$$v(0, t) = 0 \quad (t > 0), \quad v(x, 0) = -v_0 \quad (0 < x \leqslant l); \quad v_0(0) = v_0, \quad x_0(0) = 0 \tag{1.9}$$

Thus, the problem is reduced to the determination of the functions $v(x, t)$; $v_0(t)$ and $x_0(t)$, satisfying Equations (1.3), (1.7), (1.8) and (1.9).

2. The system of fundamental equations in dimensionless form. The study of impact of a visco-plastic bar on a solid body is reduced to the problem of heat conduction with a moving boundary, which is not reducible to the traditional boundary value problems of mathematical physics.

It is convenient to use the dimensionless quantities, namely

$$u(\xi, \tau) = -\frac{v(x, t)}{v_0}, \quad \xi = \frac{x}{l}, \quad \xi_0(\tau) = \frac{x_0(t)}{l}, \quad \tau = \frac{a^2 t}{l^2}, \quad u_0(\tau) = \frac{v_0(t)}{v_0} \tag{2.1}$$

Then from Equations (1.3), (1.7) to (1.9) one obtains the system of relations for determination of the unknown functions $u(\xi, \tau)$, $\xi_0(\tau)$, $u_0(\tau)$

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial \xi^2} \qquad (0 \leqslant \xi \leqslant \xi_0(\tau)) \tag{2.2}$$

$$\frac{du_0(\tau)}{d\tau} = -\frac{s}{1 - \xi_0(\tau)} \tag{2.3}$$

$$u\,[\xi_0(\tau), \tau] = u_0(\tau), \qquad \frac{\partial}{\partial \xi} u\,[\xi_0(\tau), \tau] = 0, \qquad u(0, \tau) = 0 \quad (\tau > 0) \tag{2.4}$$

$$u(\xi, 0) = 1 \quad (0 < \xi \leqslant 1), \qquad u_0(0) = 1, \qquad \xi_0(0) = 0 \tag{2.5}$$

Here $s = \sigma_0 l/\mu v_0$ is Saint Venant's parameter, which is the dimensionless combination of the known parameters and which characterized the motion.

3. Approximate solution. For an approximate solution of the system (2.2) to (2.4) the method for the boundary layer [6] given by von Karman and Pohlhausen will be used; namely we represent the function $u(\xi, \tau)$ approximately in the form*

$$u(\xi, \tau) = \begin{cases} 2u_0(\tau)\dfrac{\xi}{\xi_0(\tau)} - u_0(\tau)\dfrac{\xi^2}{\xi_0^2(\tau)} & (0 \leqslant \xi \leqslant \xi_0(\tau)) \\ u_0(\tau) & (\xi_0(\tau) \leqslant \xi \leqslant 1) \end{cases} \tag{3.1}$$

If the functions $u_0(\tau)$ and $\xi_0(\tau)$ satisfy the last two conditions of (2.4), then (3.1) satisfies all conditions of (2.4). Obviously, the function (3.1) does not satisfy Equation (2.2) exactly; it will be necessary that it satisfies the integral relation which follows from the quadrature of Equation (2.2) over the entire visco-plastic region, $(0 \leqslant \xi \leqslant \xi_0(\tau))$. Using integration by parts, and taking into account (2.4) we have

$$\int_0^{\xi_0(\tau)} \frac{\partial u}{\partial \tau} d\xi = \frac{d}{d\tau}\int_0^{\xi_0(\tau)} u d\xi - u[\xi_0(\tau), \tau]\frac{d\xi_0}{d\tau} =$$

$$= \frac{d}{d\tau}\int_0^{\xi_0(\tau)} u(\xi, \tau)\, d\xi - u_0(\tau)\frac{d\xi_0}{d\tau} = \int_0^{\xi_0(\tau)} \frac{\partial^2 u}{\partial \xi^2} d\xi = -\left(\frac{\partial u}{\partial \xi}\right)_{\xi=0}$$

Finally, we obtain the integral relation in the form

$$\frac{d}{d\tau}\int_0^{\xi_0(\tau)} u(\xi, \tau)\, d\xi - u_0(\tau)\frac{d\xi_0}{d\tau} = -\left(\frac{\partial u}{\partial \xi}\right)_{\xi=0} \tag{3.2}$$

By virtue of (3.1) we have

$$\int_0^{\xi_0(\tau)} u(\xi, \tau)\, d\xi = \frac{2}{3} u_0(\tau)\, \xi_0(\tau), \qquad \left(\frac{\partial u}{\partial \xi}\right)_{\xi=0} = \frac{2u_0(\tau)}{\xi_0(\tau)} \tag{3.3}$$

Substituting (3.3) in (3.2) and using (2.3) we obtain

$$\frac{d\xi_0}{d\tau} = \frac{6}{\xi_0(\tau)} - \frac{2s\xi_0(\tau)}{[1-\xi_0(\tau)]\, u_0(\tau)} \tag{3.4}$$

From the system of Equations (3.4) and (2.3) and using the condition (2.5) we can determine the functions $u_0(\tau)$ and $\xi_0(\tau)$; and then an approximate solution of the problem under consideration will be obtained. It is convenient to introduce new dependent variables

$$p = \frac{u_0(\tau)}{s}, \qquad q = \xi_0^2(\tau) \tag{3.5}$$

---

* This approximation coincides with that of the averaging method given by Slezkin-Targa [7].

Then, the system (3.4) and (2.3) takes the form

$$\frac{dq}{d\tau} = 12 - \frac{4q}{p(1-\sqrt{q})}, \qquad \frac{dp}{d\tau} = -\frac{1}{1-\sqrt{q}} \tag{3.6}$$

This system does not contain Saint Venant's parameter s. Consequently, the initial conditions are

$$p(0) = \frac{1}{s}, \qquad q(0) = 0 \tag{3.7}$$

Dividing the first of Equations (3.6) by the second one, it follows

$$\frac{dq}{dp} = -12(1-\sqrt{q}) + \frac{4q}{p} \tag{3.8}$$

Qualitative examination of this equation is elementary. The region ($p \geqslant 0$, $0 \leqslant q \leqslant 1$) of the integral curves given in Fig. 2 represents the solution under consideration. At the origin of the coordinate system there is a singular point of the nodal type. The integral curves emanate from the origin and have a tangent, $q = 4p$; in the neighborhood of the origin the integral curves satisfy the relation

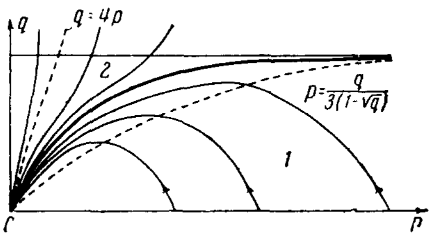

Fig. 2.

$$q = 4p + O(p^4) \tag{3.9}$$

The line of separation divides the integral curves which are emanating from the origin into two classes: the curves of Class 1 are characterized by the increasing of the ordinate q to a certain maximum, less than unity, which is on the curve $p = q/3(1 - \sqrt{q})$; further, they turn toward the abscissa intersecting it at finite points at the same angle.

In the case of integral curves of Class 2 the ordinate increases continuously, so that the curves of that class intersect the line $q = 1$ and in region ($p > 0$; $0 \leqslant q \leqslant 1$) do not return to the p-axis. Thus, the curves of that class do not intersect the p-axis at finite points.

By virtue of the initial conditions (3.7) the curves of Class 1 represent the solution of the problem; the direction of the motion for points along the integral curves with increasing time is indicated in Fig. 2 by the arrows.

4. Approximate representation of the solution for larger values of Saint Venant's parameter. Form of the bar after impact. From above examination the following qualitative deduction

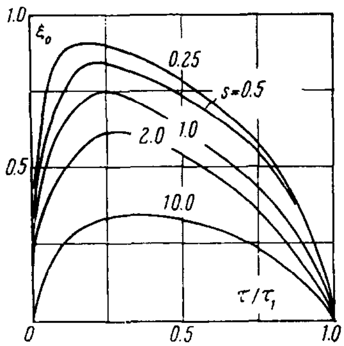

Fig. 3.

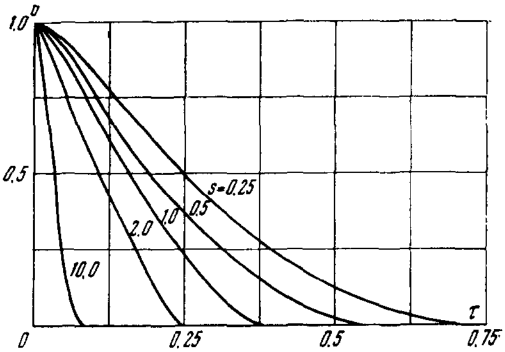

Fig. 4.

follows. At the beginning of motion the visco-plastic region is extending; its size $\xi_0(\tau) = \sqrt{(q(\tau))}$ increases, reaching its maximum at $\tau = \tau_0(s)$ (Fig. 3), and then decreases. At a certain time $\tau = \tau_1(s)$ the visco-plastic region vanishes; this instant corresponds to zero value of the velocity $u_0(\tau)$ of the elastic part of the bar (Fig. 4), so that the motion of the bar is completely stopped. Thus, in all cases a definite part of the bar joining the free boundary remains undeformed.

For a small value of τ the asymptotic representation of the basic characteristic of motion has the form

$$\xi_0(\tau) = \sqrt{12\tau} + o(\sqrt{\tau}), \qquad u_0(\tau) = 1 - s\tau \tag{4.1}$$

For τ close to τ_1, the characteristics of motion are

$$\xi_0(\tau) = 2\sqrt{\tau_1 - \tau} + o(\sqrt{\tau_1 - \tau}), \; u_0(\tau) = s(\tau_1 - \tau) \tag{4.2}$$

In the general case the system (3.6) requires numerical integration for its solution. The results of integration for a few values of Saint Venant's parameter are plotted in Figs. 3-5.

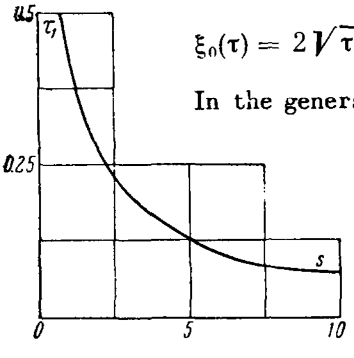

Fig. 5.

In the case of very large s, the solution can be written in explicit form. In fact, in that case, during the complete motion q is very small, so that we can neglect $\sqrt{q}$ as compared to unity, on the right-hand side of the system given by Equation (3.6). After that, the solution of system (3.6) satisfying the conditions (3.7) can be immediately written in explicit form, namely

$$q = 4\left\{\left(\frac{1}{s} - \tau\right) - s^3\left(\frac{1}{s} - \tau\right)^4\right\}, \qquad p = \frac{1}{s} - \tau, \qquad \tau_1 = \frac{1}{s} \tag{4.3}$$

In order that the solution (4.3) be applicable it is necessary that $(1 - \sqrt{q})$ differs not more from unity than, for instance, 0.1. A simple calculation shows that for such purpose s should be larger than 200. This method of calculation can be made more precise without any essential complication. We replace on the right-hand side of Equation (3.6) the factor $(1 - \sqrt{q})^{-1}$ by α and we consider α as an undetermined constant. It is very easy to see that here the solution of Equation (3.6) satisfying the condition (3.7) can be written as

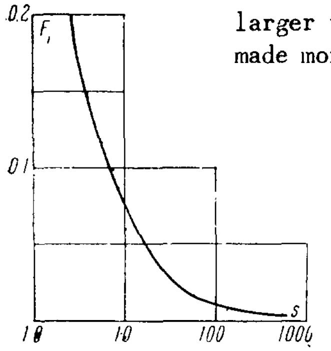

Fig. 6.

$$q = 4\left\{(\beta - \tau) - \frac{1}{\beta^3}(\beta - \tau)^4\right\}, \qquad p = \beta - \tau$$

$$\tau_1 = \beta, \qquad \beta = \frac{1}{\alpha s} \tag{4.4}$$

Now, it is necessary to establish the relation between α and the parameter s; or, which is the same, to find the relation $\beta = F(s)$. If the function $F(s)$ is known, then Formula (4.4) gives the approximate solution, where the most interesting parameters – the greatest magnitude of the visco-plastic region ξ_0^*, and the duration of the motion τ_1 – are determined by

$$\xi_0^* = \frac{1.37}{\sqrt{\alpha s}} = 1.37\sqrt{F(s)}, \qquad \tau_1 = \frac{1}{\alpha s} = F(s) \tag{4.5}$$

Thus α can be taken to be equal to the average of $(1 - \sqrt{q})^{-1}$ during the entire interval of motion. Here, the function $\beta = F_1(s)$ is defined implicitly by

$$\alpha = \frac{1}{\beta s} = \int_0^1 \frac{dy}{1 - \sqrt{4\beta(y - y^4)}}, \qquad \frac{1}{s} = \beta \int_0^1 \frac{dy}{1 - \sqrt{4\beta(y - y^4)}} \tag{4.6}$$

The plot of the function $\beta = F_1(s)$ is given in Fig. 6. For explicit analytical representation of the solution, which is sufficiently exact, we can set the constant α equal to $[1 - \overline{(\sqrt{q})}]^{-1}$, where $\overline{(\sqrt{q})}$ denotes the average value of $\sqrt{q}$ during motion. By virtue of (4.4) we

$$1 - \overline{(\sqrt{q})} = 1 - \frac{2}{\beta}\int_0^\beta \sqrt{(\beta - \tau) - \beta^{-3}(\beta - \tau)^4}\, d\tau =$$

$$= 1 - \frac{2\Gamma(3/2)\,\Gamma(1/2)}{3\Gamma(2)}\sqrt{\beta} \approx 1 - 1.05\sqrt{\beta} = \frac{1}{\alpha} = s\beta$$

Hence, the function $\beta = F_2(s)$ is determined in final form

$$\beta = F_2(s) = \frac{[\sqrt{4s + 1.05^2} - 1.05]^2}{4s^2} \tag{4.7}$$

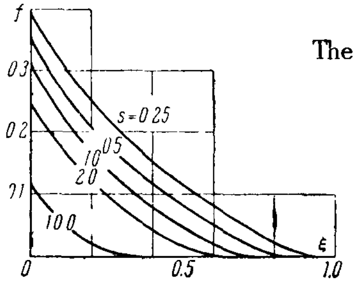

Fig. 7.

The approximate formula is sufficiently exact already for about $s > 2$.

Now we determine the form of the bar after impact. From the condition of the incompressibility of the material we have

$$F = F_0\left(1 + \frac{\partial U}{\partial x}\right)^{-1} \tag{4.8}$$

where $F = F(x)$ is the cross-section of the deformed bar; U is the instantaneous longitudinal displacement and F_0 is the cross-section of the undeformed bar. At the end of the impact, $t = t_1$, we have for an arbitrary section x

$$\frac{\partial U}{\partial x} = \int_0^{t_1} \frac{\partial v(x, t)}{\partial x}\, dt = \int_{t_*(x)}^{t_{**}(x)} \frac{\partial v(x, t)}{\partial x}\, dt = -r \int_{\tau_*(\xi)}^{\tau_{**}(\xi)} \frac{\partial u(\xi, \tau)}{\partial \xi}\, d\tau$$
$$(t_{**}(x) \geqslant t_*(x)) \tag{4.9}$$

Here $t_*(x)$ and $t_{**}(x)$ are the roots of the equation $x = x_0(t)$; $\tau_*(\xi)$ and $\tau_{**}(\xi)$ are the corresponding dimensionless quantities; $r = \rho v_0 l/\mu$ is the Reynolds number.

By virtue of (3.1) and (4.9) we find

$$\frac{\partial U}{\partial x} = -2r \int_{\tau_*(\xi)}^{\tau_{**}(\xi)} \frac{u_0(\tau)\,[\xi_0(\tau) - \xi]\, d\tau}{\xi_0^2(\tau)} = -\frac{F - F_0}{F} = -2rf(\xi) \tag{4.10}$$

Figure 7 shows, for different values of Saint Venant's parameter s, the graph $f(\xi)$ which characterizes the varying form of the bar after impact.

BIBLIOGRAPHY

1. Sokolovskii, V.V., Odnomernye nestatsionarnye dvizheniia viazko-plasticheskoi (On the uniform unsteady motion of visco-plastic media). *PMM* Vol. 13, No. 6, 1949.

2. Zverev, I.N., Rasprostranenie vozmushchenii v viazko-uprugom i viazko-plasticheskom sterzhne (Propagation of disturbances in visco-elastic and visco-plastic media). *PMM* Vol. 14, No. 3, 1950.

3. Kochetkov, A.M., Priblizhennoe reshenie nekotorykh zadach nestatsionarnogo dvizheniia viazko-plasticheskoi sredy (Approximate solution of certain problems of the unsteady motion of a visco-plastic medium). *PMM* Vol. 14, No. 4, 1950.

4. Mirzadzhanzade, A.Kh., Voprosy gidrodinamiki viazko-plasticheskikh i viazkikh zhidkostei v neftedobyche (Problems concerning visco-plasticity and viscosity in hydrodynamics of oil production). *Baku*, Aznefteizdat, 1959.

5. Lenskii, V.S., Ob uprugo-plasticheskom udare sterzhnia o zhestkuiu pregradu (On the elasto-plastic impact of a bar on a solid body). *PMM* Vol. 13, No. 2, 1949.

6. Kochin, N.E., Kibel', I.A. and Rose, N.V., *Teoreticheskaia gidromekhanika (Theoretical Hydrodynamics)*. Vol. 2, GITTL, 1948.

7. Slezkin, N.A., *Dinamika viazkoi neszhimaemoi zhidkosti (Dynamics of Viscous Incompressible liquids)*. GITTL, 1955.

Translated by M.M.S.